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A STEADY-STATE PROBLEM OF HEAT CONDUCTION IN A LAYER WITH HEAT TRANSFER CONDITIONS AT THE BOUNDARY

(STATSIONARNAIA ZADACHA TEPLOPROVODNOSTI V SLOE S USLOVIIAMI TEPLOOTDACHI NA GRANITSE)

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The following problem was investigated. A layer $(-a \leqslant z \leqslant a)$ of thickness $2a$ releases heat into the surrounding space in accordance with Newton's law,

$$\lambda \partial u/\partial n + ku = 0, \qquad z = \pm a \tag{0.1}$$

Here $\lambda (\geq 0)$ is the coefficient of heat conduction; $k (< 0)$ is the coefficient of heat transfer: the temperature of space surrounding the layer is assumed equal to zero: $\partial/\partial n$ is differentiation with respect to the exterior normal.

In the midplane of the layer $(z=0)$ lies a disk of unit radius with its center at the point $(0,0,0)$. The disk is assumed to be at the temperature

$$u|_D = g \tag{0.2}$$

It is also assumed that the function $g \in C_2$ (i.e. that it is doubly continuously differentiable). We are required to find the steady-state thermal field u in the layer without sources, i.e. the function u at all internal points of the layer (except at points on the disk) which satisfies the Laplace condition and the condition at infinity

$$\Delta u = 0; \quad u(x, y, z) \Rightarrow 0, \quad \text{for} \quad (x, y, z) \to \infty \tag{0.3}$$

The symbol $\Rightarrow$ denotes uniform convergence. In the present paper we shall find the asymptotic form of the solution for $k \to 0$ and $k \to \infty$.

The most curious case is that of the asymptotic form for $k \to 0$ (Sec. 6), which cannot be arrived at formally, and requires "nonformal" investigation of the influence function. When the layer is replaced by a bounded body, this asymptotic form can be obtained formally and can be written as

$$k^{-1}u_{-1} + u_0 + ku_1 + k^2u_2 + \ldots \tag{0.4}$$

Further, an effective computational procedure is described for the case of a thick layer and an axisymmetrical temperature distribution. Computations carried out by this method for $a = 10$ show that the asymptotic forms for $k \to 0$ and $k \to \infty$ are sufficiently conjugate. The conjugacy is accurate to within 2%. Other problems in the case of a thick layer are considered by other methods in [1 and 2].

Using the influence function which takes account of the heat transfer conditions at the layer boundary (Sec. 1), we construct a first kind integral equation (Sec. 2), find the asymptotic form of this influence function, and reduce the first kind equation to a second kind equation through inversion of the principal term of the asymptotic form.

Section 3 contains several ancillary propositions. Sections 4 to 6 give the asymptotic forms for the cases $a \to \infty$, $k \to \infty$ and $k \to 0$, respectively. Computations for evaluating the quality of conjugacy for $k \to 0$ and $k \to \infty$ in the case of large layer thickness $2a$ are carried out in Section 7. In Section 8 we justify the asymptotic form of the influence function for $k \to 0$ given without proof in Section 1.

1. The influence function can be found with the aid of the Hankel transform and written as

$$G(x_0, y_0; x, y, z) = \frac{1}{r} + \frac{1}{a}\int_0^\infty \frac{(t-\nu)e^{-t}}{\nu\cosh t + t\sinh t}\cosh\left(\frac{z}{a}t\right)J_0\left(\frac{\rho}{a}t\right)dt \qquad (1.1)$$

$$r = \sqrt{(x-x_0)^2+(y-y_0)^2+z^2}, \qquad \rho = \sqrt{(x-x_0)^2+(y-y_0)^2}, \qquad \nu = ak/\lambda$$

Here J_0 is a zeroth-order Bessel function of the first kind. The function G is the temperature at the point (x, y, z) produced by a heat source of unit power situated at the point $(x_0, y_0, 0)$ in the layer releasing heat through its boundaries in accordance with law (0.1).

In order to consider the case $a \to \infty$ we make use of the expansion (1.2)

$$G(x_0, y_0; x, y, 0) = \frac{1}{r} + \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\frac{r^{2n}}{(n!)^2 2^{2n} a^{2n}} I_\nu(n), \qquad I_\nu(n) = \int_0^\infty \frac{(t-\nu)e^{-t}t^{2n}}{\nu\cosh t + t\sinh t}dt$$

Series (1.2) is obtained by replacing the function J_0 by its Taylor series. Clearly, this series (together with its derivatives with respect to x and y) converges uniformly in x, y and ν when $r/a \leq q < 1$. In investigating the asymptotic form for $k \to \infty$ ($\nu \to \infty$) we make use of the asymptotic expansion

$$G(x_0, y_0; x, y, 0) \approx \frac{1}{r} - \frac{1}{a}\int_0^\infty \frac{e^{-t}}{\cosh t}J_0\left(\frac{r}{a}t\right)dt -$$

$$-\frac{1}{a}\sum_{n=1}^{\infty}\left(\frac{-1}{\nu}\right)^n\int_0^\infty \frac{t^n e^{-t}}{\cosh t}J_0\left(\frac{r}{a}t\right)(\tanh t)^{n-1}(1+\tanh t)\,dt \qquad (1.3)$$

This expansion is not a convergent series, but merely yields an asymptotic representation for $\nu \to \infty$ with uniform estimates (and for derivatives of all orders with respect to x and y) when the quantity r is bounded above and a from zero below.

Formula (1.3) can be obtained as follows. The integral in (1.1) breaks down into two: one with limits $(0, \nu/2)$, and the other with $(\nu/2, \infty)$; the latter is uniformly estimable as $O(e^{-\nu/4})$. We can then represent the denominator of the first integral as $\nu\cosh x$ $(1 + x(\tanh x)/\nu)$ and then apply the geometric progression formula; the next step is to return to an integral with limits $(0, \infty)$ in each term.

It should be noted that if we were considering a bounded domain rather than a layer, the influence function would have turned out to be a convergent series.

For $k \to 0 (\nu \to 0)$ the expansion is of the form

$$G(x_0, y_0; x, y, 0) = \frac{1}{r} - \frac{1}{2a} \ln \nu + \ln \nu \sum_{i=1}^{\infty} \nu^i K_i \left(\frac{r}{a}\right) + \sum_{i=0}^{\infty} \nu^i L_i \left(\frac{r}{a}\right) \qquad (1.4)$$

The series converges for sufficiently small ν. This convergence is uniform (and applies to all derivatives of the series with respect to x and y) when the quantity r is bounded above and the quantity a below from zero. The expansion is justified in Section 8. Formula (1.4) enables us to draw the following conclusion.

For a given source distribution over the disk, the temperature increases without limit as $k \to 0$.

The principal increasing part depends solely on the integral power N and is of the form $(1/2a)N \ln \nu$.

The first part of the derivation is physically clear, since in the limiting case $k = 0$ thermal energy does not flow out of the layer, but accumulates in it, making a steady-state impossible.

It should be noted that in the case of a bounded body the influence function expands into a convergent series in power of ν, the expansion starting with $1/\nu$.

2. The solution of the problem can be sought in the form

$$u(x, y, z) = \iint_D G(x_0, y_0; x, y, z) f(x_0, y_0) dx_0 dy_0 \qquad (2.1)$$

where f is the source density which is a function summable with the power $p \ (1 < p < 2) \ (f \in L_p \ (D))$.

We then obtain the following integral equation of the first kind for f:

$$\iint_D G(x_0, y_0; x, y, 0) f(x_0, y_0) dx_0 dy_0 = g(x, y), \qquad x^2 + y^2 \leqslant 1 \qquad (2.2)$$

Eq. (2.2) and problem (0.1)-(0.4) are related in the following way. They are both solvable and have unique solutions. The solution u of the problem is expressible in terms of the solution of the integral equation by way of Formula (2.1); the reverse relationship is given by Formula

$$f(x_0, y_0) = -\frac{1}{2\pi} \frac{\partial u(x_0, y_0, 0)}{\partial z} \qquad x_0^2 + y_0^2 < 1 \qquad (2.3)$$

We shall formulate four ancillary assertions which will be of use later on; the first Lemma reflects the maximum principle in the case of a nonconstant temperature of the ambient medium; the second Lemma is the physically obvious fact of monotonous temperature decrease with a decreasing coefficient of heat transfer; the third and fourth Lemmas are mathematical consequences of the first two.

3. Let us cite the ancillary assertions.

Lemma 3.1. Let the function u satisfy conditions (0.2) to (0.4) and the condition

$$\lambda \, \partial u / \partial n + k(u - \varphi) = 0, \quad z = \pm a$$

where φ is a specified function on the planes $z = \pm a$. It is assumed that the function φ is continuous and that $\varphi(x, y, \pm a) \to 0$ as $(x, y) \to \infty$. Then

$$\min [\min g, \min \varphi] \leqslant u \leqslant \max [\max g, \max \varphi]$$

The proof is analogous to that of the ordinary maximum principle.

Lemma 3.2. The solution of problem (0.1)-(0.4) does not decrease if $g \geq 0$ and $k \downarrow 0$.

Proof. We consider the two solutions u_1 and u_2 corresponding to the coefficients $k = k_1$ and k_2, respectively. Let $k_2 > k_1$. Then

$$0 = \lambda \frac{\partial (u_1 - u_2)}{\partial n} + k_1 \left[(u_1 - u_2) - \frac{k_2 - k_1}{k_1} u_2 \right]$$

at the planes $z = \pm a$. At the disk D we have $u_2 - u_1 = 0$. By Lemma 3.1, $u_2 \geq 0$, so that (by the same Lemma), $u_1 - u_2 \geq 0$. Lemma 3.2 has been proved.

Lemma 3.3. In any bounded closed domain not containing the disk D and the planes $z = \pm a$, the solution of problem (0.1)-(0.4) and all of its derivatives tends to some limiting function as $k \to 0$.

This Lemma follows from the preceding one.

Lemma 3.4. The normal derivative $\partial u / \partial z$ of the solution of problem (0.1)-(0.4) is summable over the disk with any power $(1 <) p (< 2)$ and converges in the same norm for $k \to 0$. Further, we have the estimate $\| \partial u / \partial n \|_p \leqslant A \| g \|_{C_2}$, where A is a constant independent of k.

Note. This Lemma remains valid in the case $a = \infty$. Here condition (0.1) becomes the condition $u(\infty) = 0$ and the solution is independent of k.

The proof of Lemma 3.4 is exceedingly cumbersome and is therefore omitted.

4. The case $a \to \infty$. Expansion (1.2) has the following useful properties. All the partial sums of the infinite series have degenerate kernels, i.e. they can be represented as the finite sum $\Sigma a_k(x, y)\, b_k(x, y)$. Inverting the side containing the kernel $1/r$, we arrive at the equation of the second kind:

$$f = \frac{1}{a} Sf + G_0^{-1} g, \qquad S = \sum_{n=0}^{\infty} \left(\frac{1}{a} \right)^{2n} G_0^{-1} H_n \tag{4.1}$$

Here G_0^{-1} is the inversion of the operator G_0; the latter operator and the operator H_n are given by Formulas

$$G_0 f = \iint_D \frac{f(x_0, y_0)}{r} dx_0\, dy_0, \qquad H_n f = \iint_D \left[(-1)^n I_\nu(n) \frac{r^{2n}}{n! \, 2^{2n}} \right] f \, dx_0\, dy_0$$

Since the integral equation $G_0 f = g$ is related to the Dirichlet problem for the exterior of a disk in unbounded space, from the note following Lemma 3.4 we conclude that the series S converges for $a > 1$ as a series of operators acting from L_p into L_p $(1 < p < 2)$. This enables us to draw the following conclusions.

1. For sufficiently large a the solution of Eq. (4.1) can be expanded into the series

$$f = f_0 + a^{-1} f_1 + a^{-2} f_2 + a^{-3} f_3 + \ldots \tag{4.2}$$

which converges in the norm of the space L_p.

A similar expansion is valid for the temperature field u; this expansion converges in the norm of the space C (of continuous functions with the norm $\| u \|_C = \max_D | u |$).

2. The terms of this series can be computed by solving truncated Eq. (4.2). This can be readily solved as an equation with a degenerate kernel provided one has a good expression for the operator G_0^{-1}. In the axisymmetrical case such an expression exists (e.g. see [3], p. 423).

Section 7 contains computations for $g = 1$. We note the following fact which facilitates computations: in order to obtain N terms of series (4.2) it is necessary to retain $[N/2]$ terms in Eq. (4.1).

5. The case $k \to \infty$ ($\nu \to \infty$). Here we make use of expansion (1.3) whose principal part can be written as

$$G_0 = \frac{1}{r} - \frac{1}{a}\int_0^\infty \frac{e^{-t}}{\cosh t} J_0\left(\frac{r}{a}t\right) dt$$

and is an influence function (on the plane $z = 0$) of a unit source in a layer at zero temperature at the boundary $z = \pm a$. Inverting the integral operator corresponding to the principal part and making use of Lemma 3.4, we arrive at an equation of the second kind of the form

$$(I + M) f = G_0^{-1} g, \qquad G_0^{-1} g \in L_p \qquad (1 < p < 2)$$

Here I is an identity operator; M is a completely continuous operator which acts from L_p into L_p ($1<p<2$) and is expandable in the asymptotic series

$$M \approx \sum_{n=1}^\infty \left(\frac{1}{\nu}\right)^n M_n \quad \text{or} \quad \left\| M - \sum_{n=1}^N \left(\frac{1}{\nu}\right)^n M_n \right\|_{L_p \to L_p} = O\left(\frac{1}{\nu^{N+1}}\right)$$

This enables us to draw the following conclusion.

The source density and the temperature field u as $k \to \infty$ can be represented as the asymptotic series

$$f \approx \sum_{n=0}^\infty \left(\frac{1}{\nu}\right)^n f_n, \qquad u \approx \sum_{n=0}^\infty \left(\frac{1}{\nu}\right)^n u_n$$

The asymptotic series for f should be interpreted in the sense of the space L_p ($1 < p < 2$); in the case of u it should be interpreted in the sense of the space C. It should be noted that in the case of a bounded domain with heat transfer conditions the analogous series would be convergent.

6. The case $k \to 0$ ($\nu \to 0$). Here we make use of expansion (1.4). In contrast to the preceding cases, the influence function increases without limit as $k \to 0$. The increasing part of $\ln \nu/2a$ remains constant, however. This fact will prove useful below. We proceed with our analysis as follows. First, we reduce Eq. (2.2) to an equation of the second kind by inverting the operator G_0 with the kernel $1/r$,

$$f - \frac{\ln \nu}{2a}(G_0^{-1} 1)\iint_D f(x_0, y_0)\, dx_0\, dy_0 + \left(\sum_{k=1}^\infty \nu^k \ln \nu\, G_0^{-1} K_k\right) f + \left(\sum_{k=0}^\infty \nu^k G_0^{-1} L_k\right) f = G_0^{-1} g \tag{6.1}$$

where K_k and L_k are integral operators corresponding to the kernels K_k and L_k. The operators $G_0^{-1} K_k$ and $G_0^{-1} L_k$ act completely continuously from L_p into L_p ($1<p<2$), while their operator series converge absolutely for sufficiently small ν(*).

*) The series consisting of the operators B_k converges absolutely if the $\|B_k\|$ series converges.

Now, eliminating the increasing part, we invert the operator $I - \ln\nu K/2a$, where K is an operator of the form

$$Kf = (G_0^{-1}1)\iint_D f(x_0, y_0)\,dx_0\,dy_0$$

The inverse operator R is of the form

$$R\psi = \left(I - \frac{\ln\nu}{2a}K\right)^{-1}\psi = \psi + \left[2a - \ln\nu\iint_D (G_0^{-1})\,dx\,dy\right]^{-1}\ln\nu\,(G_0^{-1})\iint_D \psi\,dx\,dy =$$

$$= R_0\psi - \varphi(\nu, a)R_1\psi$$

$$R_0\psi = \psi - \left[\iint_D (G_0^{-1}1)\,dx\,dy\right]^{-1}(G_0^{-1}1)\iint_D \psi\,dx\,dy$$

$$\varphi(\nu, \alpha) = \sum_{k=1}^{\infty}\left[\frac{1}{2a}\ln\nu\iint_D (G_0^{-1})\,dx\,dy\right]^{-k},\quad R_1\psi = \left[\iint_D (G_0^{-1}1)\,dx\,dy\right]^{-1}(G_0^{-1})\iint_D \psi\,dx\,dy$$

After inversion Eq. (6.1) becomes

$$f + R_0G_0^{-1}L_0f + \ln\nu\left(\sum_{k=1}^{\infty}\nu^kR_0G_0^{-1}K_k\right)f + \left(\sum_{k=1}^{\infty}\nu^kR_0G_0^{-1}L_k\right)f -$$
$$-\varphi(\nu,\alpha)\ln\nu\left(\sum_{k=1}^{\infty}\nu^kR_1G_0^{-1}K_k\right)f - \varphi(\nu, a)\left(\sum_{k=0}^{\infty}\nu^kR_1G_0^{-1}L_k\right)f =$$
$$= R_0G_0^{-1}g - \varphi(\nu, a)R_1G_0^{-1}g \tag{6.2}$$

All of the operators in the left-hand side are considered in L_p ($1 < p < 2$). This equation has the following properties:

1) For $g = 1$ we have $R_0G_0^{-1}g = 0$, so that the right-hand side tends to zero together with $\varphi(\nu, a)$.

2) The equation

$$R_0G_0^{-1}g = G_0^{-1}g, \qquad \text{if}\quad \iint_D (G_0^{-1}g)\,dx\,dy = 0$$

is valid

3) For any function $g\ (\in C_2)$

$$\iint_D (R_0G_0^{-1}g)\,dx\,dy = 0$$

4) The operators in all the terms except the first two tend to zero in their norm as $k \to 0$. The first two terms are independent of k.

5) The operator $I + R_0G_0^{-1}L_0$ (I is an identity operator) is invertible.

The first four properties are self-evident. The proof of property (5) must be prefaced by the following ancillary propositions:

a) Any infinitely differentiable function ψ which together with all its derivatives vanishes at the edge of disk and for which

$$\iint_D \psi\,dx\,dy = 0$$

belongs to the image of the operator $I + R_0G_0^{-1}L_0$. For our proof let us take $g = G_0\psi$. Then $g \in C_2$, and by virtue of property (2) we have $R_0G_0^{-1}(G_0\Psi) = \psi$. For $k>0$ Eq. (6.2) has a solution f which as $k\to 0$ converges in L_p $(1<p<2)$ to some function f_0 (see Lemma 3.4). Taking the limit in Eq. (6.2), we obtain $f_0 + R_0G_0^{-1}L_0f_0 = \psi$ (see property (4)).

b) The image of the operator $I + R_0G_0^{-1}L_0$ contains the function

$$\psi_0 = 1 + R_0G_0^{-1}L_0 1, \qquad \psi_0 \in L_p, \qquad \iint_D \psi_0\, dx\, dy \neq 0$$

Proof of property (5). Functions of the form $\psi + \alpha\psi_0$ (ψ, ψ_0 satisfy the conditions of statements a), b), respectively) form a set which is everywhere dense in the space L_p and belong to the image. The image of the operator under investigation is closed, since the operator satisfies the Fredholm theorem (see [4]). Hence, the image coincides with the entire space L_p, and the operator is invertible. Property (5) has been proved.

Action of the operator $B = (I + R_0G_0^{-1}L_0)^{-1}$ on Eq. (6.2) yields an equation of the form

$$f - \Big(\ln\nu \sum_{k=1}^{\infty} \nu^k K_k' + \sum_{k=1}^{\infty} \nu^k L_k' - \varphi(\nu, a)\ln\nu \sum_{k=1}^{\infty} \nu^k K_k'' - \varphi(\nu,a)\sum_{k=1}^{\infty} \nu^k L_k''\Big) f = \psi, \quad \psi = BR_0G_0^{-1}g - \varphi(\nu, \alpha)\, BR_1G_0^{-1}g \qquad (6.3)$$

Here K_k', K_k'', L_k', L_k'' are linear operators acting from L_p into L_p.

All of the operator series converge absolutely for sufficiently small $k(\nu)$, and the operator in parentheses is smaller than unity in norm for sufficiently small k. The latter fact can be established by an estimate involving the substitution of operators in the infinite sums by their norms. From the foregoing it follows that the function f can be expresses as the infinite sum

$$f = \psi + (\ )\,\psi + (\ )^2\psi + \cdots + (\ )^n\,\psi + \cdots$$

The symbol ( ) represents the operator appearing in parentheses in (6.3). By virtue of absolute convergence, the power of the series arising when the expressions in parentheses are raised to their respective powers can be represented as multiple series. Expanding the function $\varphi(\nu, a)$ (which also appears in ψ) in a series in power of $1/\ln\nu$, collecting like terms, and arranging them in decreasing order in ν, we can obtain the following conclusions.

For sufficiently small k the source density f can be expanded in the double series

$$f = \sum_{n=0}^{\infty} \nu^n \sum_{i=-n}^{\infty} \left(\frac{1}{\ln\nu}\right)^i f_{ni}, \quad \left(\iint_D f_{00}\, dx\, dy = 0\right) \qquad (6.4)$$

which converges in the norm of the space L_p $(1<p<2)$.

If $g = 1$, then $R_0G_0^{-1}g = 0$ and $\psi = 0$; thus $f_{00} = 0$ in expansion (6.4).

Hence, for $g = 1$ the density f diminishes with k as $1/\ln k$.

For the temperature field u we have an expansion similar to (6.4), with the difference that convergence in the former case must be understood in the sense of the space C.

7. Let us demonstrate the computational procedure proposed in Section 4 for large a in the case $g = 1$, as well as the conjugacy of the expansion for $k\to\infty$ and $k\to 0$. We

retain one term in the expansion of (1.2) in power of $1/a$ (the remaining portion is of order a^{-3}). The resulting truncated integral equation for determining the density is of the form

$$\iint\limits_D \frac{f(x_0, y_0)\,dx_0\,dy_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 1 - \frac{1}{a}\iint\limits_D T_0(\nu) f(x_0, y_0)\,dx_0\,dy_0, \qquad T_0(\nu) = \int_0^\infty \frac{(t-\nu)\,e^{-t}\,dt}{\nu\cosh t + t\sinh t}$$

The value of the density f thus determined differs from the true value (in the norm of L_p $(1<p<2)$) by a quantity on order a^{-3}.

We denote the total power of the disk sources by N; the function f can be expressed in terms of N,

$$f(x, y) = \frac{1}{\pi^2}\left(1 - \frac{1}{a}\;T_0(\nu)\,N\right)\frac{1}{\sqrt{1-x^2-y^2}}$$

After integration, we have the following formula for the quantity N:

$$N = \frac{2}{\pi}\left[1 + \frac{2}{\pi}\,\frac{1}{a}\,T_0(\nu)\right]^{-1} \tag{7.1}$$

In order to investigate the question of the goodness of conjugacy of the asymptotic forms we must compute the corresponding representation (ζ is the Riemann function):

For $k\to\infty$

$$T_0(\nu) \approx -\ln 2 + (\ln 2)\,\nu^{-1} - (\ln 2)\,\nu^{-2} + [\ln 2 + {}^3/_8\,\zeta(3)]\,\nu^{-3} + \ldots \tag{7.2}$$

For $k\to 0$

$$T_0(\nu) = -{}^1/_2\,\ln\nu - 0.69315 - 0.16667\,\nu\ln\nu + 0.40486\,\nu + 0.03333\,\nu^2\ln\nu + \ldots \tag{7.3}$$

Substituting Expressions (7.2), (7.3) into Formula (7.1), we obtain asymptotic representations for N. Further computations were carried out for $a = 10$ by means of the Formulas

$$N \approx 0.6659 - 0.0306\,\nu^{-1} + 0.0318\,\nu^{-2} \qquad (k\to\infty) \tag{7.4}$$

$$N \approx 2\,(-0.1\ln\nu + 3.0030 - 0.0333\,\nu\ln\nu + 0.0810\,\nu + 0.0067\,\nu^2\ln\nu + 0.0027\,\nu^2)^{-1} \tag{7.5}$$

Some of the results obtained are as follows:

$\nu =$	0.1	0.2	0.4	0.6	0.8	1	2	2.5	3
$N(k\to\infty) =$	—	—	—	0.703	0.677	0.667	0.659	0.659	0.659
$N(k\to 0) =$	0.616	0.627	0.637	0.643	0.646	0.648	0.650	0.647	0.642

Clearly, the disparities between the numbers obtained from Formulas (7.4) and (7.5) constitute less than 2% over a significant range of ν values ($1.5 \leqslant \nu \leqslant 2.5$). It should be expected that such is always the case with sufficiently large a. With small a, certain objections can be raised against this statement. In fact, the larger the thickness a, the smaller the role played by conditions at the boundary of the layer (provided the temperature field is considered in a fixed neighborhood of the disk). This may explain the good conjugacy, although boundary conditions can play a decisive role with small thicknesses.

8. We shall now derive the asymptotic form of influence function (1.4) for $k\to 0$ ($\nu\to 0$).

The integral in Formula (1.1) breaks down into two,

$$\int_0^\infty \frac{(t-\nu)e^{-t}}{\nu\cosh t + t\sinh t} J_0\left(\frac{r}{a}t\right) dt = \int_0^c \cdots + \int_c^\infty \cdots \tag{8.1}$$

Here c is a positive number which will be chosen below.

The second integral is clearly expandable in a convergent series in powers of ν,

$$\int_c^\infty \frac{(t-\nu)e^{-t}}{\nu\cosh t + t\sinh t} J_0\left(\frac{r}{a}t\right) dt = \int_c^\infty e^{-t} J_0\left(\frac{r}{a}t\right)\frac{dt}{\sinh t} +$$

$$+\sum_{k=1}^\infty (-\nu)^k \int_c^\infty \left[\left(\frac{\coth t}{t}\right)^k + \frac{1}{t}\left(\frac{\coth t}{t}\right)^{k-1}\right] J_0\left(\frac{r}{a}t\right) e^{-t}\frac{dt}{\sinh t} \qquad \left(\nu < \frac{c}{\coth c}\right) \tag{8.2}$$

We subject the first integral to the following transformations:

$$\int_0^c \frac{(t-\nu)e^{-t}}{\nu\cosh t + t\sinh t} J_0\left(\frac{r}{a}t\right) dt = -\int_0^c J_0\left(\frac{r}{a}t\right) dt + \int_0^c \frac{t\cosh t + \nu\sinh t}{\nu\cosh t + t\sinh t} J_0\left(\frac{r}{a}t\right) dt$$

In the second integral we introduce the new integration variable τ which is related to the Eq. $\nu\cosh t + t\sinh t = \nu + \tau^2$.

We now choose the constant c in such a way that the function $t(\tau)$ can be expanded in a convergent Taylor series for $|\tau| \le 4c\sinh c$. It is clear that $t(\tau)$ turns out to be an odd function; hence, $t'(\tau)$ is an even function.

Carrying out one more substitution of variables $\tau^2 = \zeta$ and taking account of the evenness of the integrand, we reduce the second integral to the form

$$\int_0^m \frac{\varphi(\zeta)\,d\zeta}{\zeta+\nu}$$

where $\varphi(\zeta)$ can be expanded into a convergent series for $|\zeta| < 2m$.

Further, the last integral can be transformed as follows:

$$\int_0^m \frac{\varphi(\zeta)}{\zeta+\nu} d\zeta = \int_0^m \frac{\varphi(\zeta)-\varphi(-\nu)}{\zeta+\nu} d\zeta + \varphi(-\nu)\left[\ln(m+\nu) - \ln\nu\right] \tag{8.3}$$

This integral expands in a series in powers of ν which converges for $\nu < m$. It is easy to verify that the requirements of uniform convergence noted in expansion (1.4) are also fulfilled. This completes our justification of expansion (1.4) of Section 1.

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ON THE KÁRMÂN-HOWARTH EQUATION

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Let us take the two points x, y, z and x, y', z in a Cartesian coordinate system. Let the nonsimultaneous velocity components at these points be $u(t)$, $v(t)$, $w(t)$ and $u'(t')$, $v'(t')$, $w'(t')$, respectively.

We can then write the Kármán-Howarth equation in the case of homogeneous isotropic turbulence in two ways

$$\frac{\partial}{\partial t}\langle vv'\rangle + \left[\frac{\partial}{\partial r} + \frac{4}{r}\right]\langle u^2v'\rangle = \nu\left[\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right]\langle vv'\rangle$$

$$\frac{\partial}{\partial t'}\langle vv'\rangle - \left[\frac{\partial}{\partial r} + \frac{4}{r}\right]\langle u'^2v\rangle = \nu\left[\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right]\langle vv'\rangle$$

Here

$$r = y' - y, \quad \langle vv'\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} v_n v'_n \quad (n \text{ is the number of the experiment})$$

$$\langle u^2v'\rangle = f(r, t, t'), \qquad \langle u'^2v\rangle = -f(r, t', t)$$

These equations are independent, form a closed system, and permit elimination of the second moments. It follows that

$$\left[\frac{\partial}{\partial r} + \frac{4}{r}\right] f(r, t, t') = F(r, t, t')$$

$$\frac{\partial}{\partial t'}F(r, t, t') - \frac{\partial}{\partial t}F(r, t', t) = \nu\left[\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right]\left[F(r, t, t') - F(r, t', t)\right]$$

is the functional differential equation in the third moments.

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